MATH 161 - Fall 2013

Lecture 3

Lecture 3 08/30/2013

## THE LIMIT OF A FUNCTION

### MOTIVATION

**Example** Spse that a ball is dropped from a hill 1000 m above the ground

Find the velocity of the ball after 5 secs ??

Soln If the distance travelled after t secs is denoted by $s(t)$ and measured in meters, then it is given by $s(t) = 4.9t^2$

The trouble we have is that we need to find velocity at a specific time $t = 5s$ rather than a time interval.

However we can approximate the velocity at $t = 5s$ by computing the average velocity over the brief time interval $t = 5$ to $t = 5.1$

$$\text{Average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5.1) - s(5)}{5.1 - 5} = \frac{4.9(5.1)^2 - 4.9(5)}{0.1}$$

$$= 49.49$$

| Time Interval | Average velocity |
|---|---|
| $5 \leq t \leq 6$ | 53.9 |
| $5 \leq t \leq 5.1$ | 49.49 |
| $5 \leq t \leq 5.05$ | 49.245 |
| $5 \leq t \leq 5.01$ | 49.049 |
| $5 \leq t \leq 5.001$ | 49.0049 |

Average velocity
over successively smaller
time periods.

So as we make time period shorter, the average velocity is becoming closer to 49 m/s

The instaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter periods that start $t = 5$

The velocity after 5s is 49m/s.

INTUITIVE DEFINITION OF LIMIT

Lets investigate the behaviour of the function $f$ defined by $f(x) = x^2 - 2$ for values of $x$ near 2.

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| x | f(x) | x | f(x) |
|---|---|---|---|
| 1.8 | 1.24 | 2.2 | 2.84 |
| 1.9 | 1.61 | 2.1 | 2.41 |
| 1.99 | 1.9601 | 2.01 | 2.0401 |
| 1.999 | 1.996001 | 2.001 | 2.004001 |

We see that as x is close to 2 from either side, f(x) is close to 2.

So actually make the values of f(x) as close as we like to 2 by taking x sufficiently close to 2.

We express this by saying "the limit of the function $f(x) = x^2 - 2$ as x approaches 2. The notation is

$$\lim_{x \to 2} (x^2 - 2) = 2$$

DEFN Spse f(x) is defined when x is near number a. Then we write

$\lim_{x \to a} f(x) = L$ and say " limit of f(x), as x approaches a, equals L "

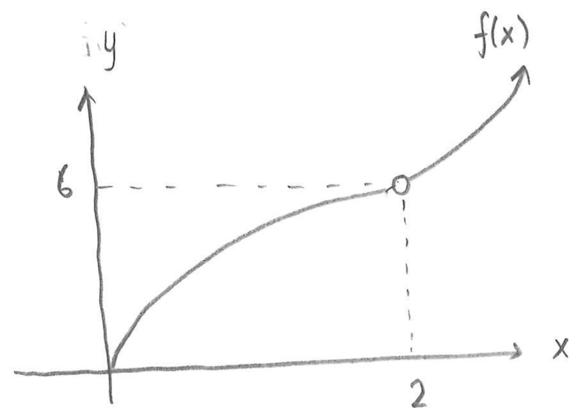
If we can make the values of f(x) as close to L as we like, by taking x to be sufficiently close to a, but not equal to a.

We also say " f(x) approaches L as x approaches a "

IMP When finding the limit of $f(x)$ as $x$ approaches $a$, we never consider $x = a$.

As a matter of fact, we don't even need $f(x)$ to be defined at $x = a$.

The only thing that matters is how $f$ is defined near $a$.



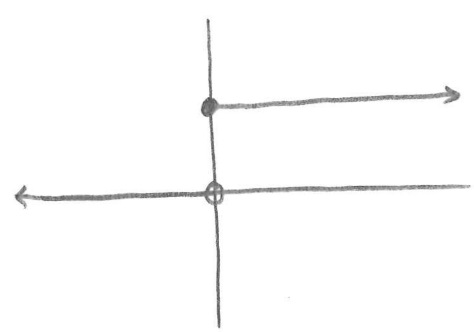

HOLE INDICATES THAT EXACT POINT IS MISSING

$$\lim_{x \to 2} f(x) = ?$$

$\lim_{x \to 2} f(x) = 6$ even though $f(x)$ is not defined at $x = 2$.

Ex <u>THE HEAVISIDE FUNCTION</u>

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



So what is $\lim_{x \to 0} H(t)$ ??

As you approach from left as $x$ approaches $0$, $H(t)$ approaches $0$.

As you approach from the right to $0$, $H(t)$ approaches $1$

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So there is no single number that $H(t)$ approaches as $t$ approaches 0

Therefore $\lim_{t \to 0} H(t)$ does not exist.

## ONE-SIDED LIMITS

In the above example, $H(t)$ approaches 0 as $t$ approaches 0 from the left

and $H(t)$ approaches 1 as $t$ approaches 0 from the right.

We indicate this situation symbollically by writing,

$$\lim_{t \to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} H(t) = 1$$

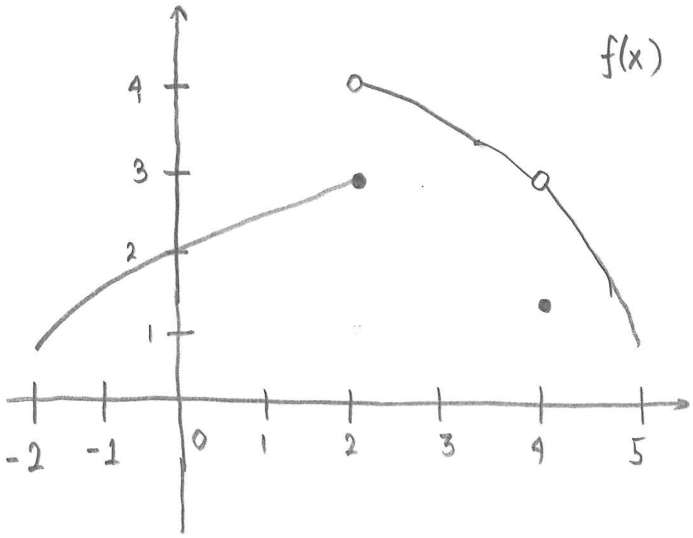
The symbol "$t \to 0^-$" indicates that we consider only values of $t$ that are less than 0.

Likewise, "$t \to 0^+$" indicates " " " " " " " " " greater than 0.

### Definition

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example




a) $\lim_{x \to 2^+} f(x) = 4$

b) $\lim_{x \to 2^-} f(x) = 3$

c) $\lim_{x \to 2} f(x) = \text{DNE}$

d) $\lim_{x \to 4^-} f(x) = 3$

e) $\lim_{x \to 4^+} f(x) = 3$

f) $\lim_{x \to 4} f(x) = 3$

g) $f(4) = 1$

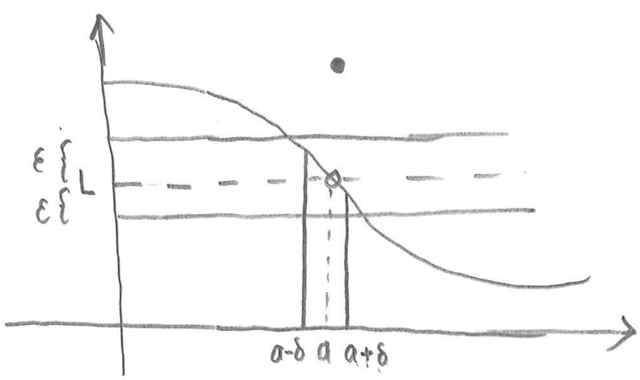
## PRECISE DEFINITION OF A LIMIT

Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then we say

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that if

$$0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon .$$




So start with a given $\varepsilon > 0$, draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of $f$. If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict $x$ to lie in the interval $(a-\delta, a+\delta)$ and take $x \neq a$ then the curve lies between $y = L - \varepsilon$ and $y = L + \varepsilon$

If time permits

$$f(x) = \begin{cases} x+3, & x \leq +2 \\ 6, & +2 < x < 3 \\ x^2 - 3, & x \geq 3 \end{cases}$$

a) $\lim_{x \to 2^-} f(x)$

b) $\lim_{x \to 2^+} f(x)$

c) $\lim_{x \to 2} f(x)$

d) $\lim_{x \to 3^-} f(x)$

e) $\lim_{x \to 3^+} f(x)$

f) $\lim_{x \to 3} f(x)$